Information Theory Back paper
(3 hours, Dec 2024)

Use a separate sheet for each problem; write your roll number at the top of each page. No book, no notes, no use of the internet. Attempt all problems.

Name: ____________________

Roll number: ____________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (a) Suppose C is a prefix-free code over the alphabet $\{0, 1, a, b, c\}$ with n_i codewords of length i. Assume that we use $1, 2, 3, \ldots$ as indices for the positions in the codewords. Suppose all codewords in C have a letter from $\{0, 1\}$ in the odd positions, and a letter from $\{a, b, c\}$ in the even positions. Show that $\boxed{10}$

$$\sum_{i \geq 0} \frac{n_i}{2^{\lceil i/2 \rceil} \cdot 3^{\lfloor i/2 \rfloor}} \leq 1,$$

where $\lfloor x \rfloor$ is the greatest integer that is at most x and $\lceil x \rceil$ is the smallest integer that is at least x. [Hint: Arrange the codewords in an appropriate tree and consider a random walk starting at the root.]

(b) Suppose $A = \{a_1, a_2, a_3, a_4, a_5\}$. Construct an optimal (with minimum expected length) prefix-free encoding for A over the alphabet $\{0, 1\}$ if the probabilities of the symbols in A are $p(a_1) = 0.35$, $p(a_2) = 0.1$, $p(a_3) = 0.15$, $p(a_4) = 0.2$, and $p(a_5) = 0.2$. [You may represent the encoding as a binary tree; it is not essential to list the codewords.] $\boxed{10}$

2. Suppose A and B are finite non-empty sets. Let X, Y be random variables taking values in the set $A \times B$. Let $p_{XY}(a, b) = \Pr[X = a \text{ and } Y = b]$, $p_X(a) = \Pr[X = a]$ and $p_Y(b) = \Pr[Y = b]$. Recall that $I[X : Y] = H[X] + H[Y] - H[XY]$ and $H[Y|X] = H[(X, Y)] - H[X]$.

(a) Show that $\boxed{10}$

$$I[X : Y] = \sum_{a,b} p_{XY}(a, b) \log_2 \frac{p_{XY}(a, b)}{p_X(a) p_Y(b)}.$$

State why $I[X : Y] \geq 0$.

(b) Conclude that $H[Y|X] \leq H[X]$. $\boxed{5}$

(c) State why $H[Y|X] \geq 0$. $\boxed{5}$

3. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$. Consider a source that emits a pair of symbols $(X, Y) \in A \times B$, where X and Y are independent. Suppose $\Pr[X = a_1] = 1/2$, $\Pr[X = a_2] = 1/3$ and $\Pr[X = a_3] = 1/6$; suppose $\Pr[Y = b_1] = 1/6$, $\Pr[Y = b_2] = 1/3$, $\Pr[Y = b_3] = 1/3$ and $\Pr[Y = b_4] = 1/6$. Suppose (X^n, Y^n) denotes n iid samples drawn from this source; for a set $S \subseteq (A \times B)^n$, let $P(S) = \Pr[\langle (X_1, Y_2), (X_2, Y_2), \ldots, (X_n, Y_n) \rangle \in S]$. Let $\boxed{20}$

$$R = \lim_{n \to \infty} \frac{1}{n} \min_{S \subseteq (A \times B)^n : P(S) \geq \frac{1}{2}} \log_2 |S|.$$

What is R?

4. Consider a channel $\mathcal{C}$ with input alphabet A and output alphabet B. Let $p_{b|a}$ denote the probability that the output of the channel is $b \in B$ when the input is $a \in A$.

(a) Recall that the capacity of this channel is given by the following expression $\boxed{10}$

$$\mathsf{cap}(\mathcal{C}) = \max_{(X:Y):?} I[X : Y],$$

where (X, Y) are random variables taking values in $A \times B$ with joint distribution satisfying some condition denoted by "?". What is this condition?

(b) Suppose the characteristics of the channel $\mathcal{C}$ are given by the following matrix $\boxed{10}$

$$M = (m_{ij}) = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \end{pmatrix};$$

that is, $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$ and the ij-th entry $m_{ij} = p(j|i)$. Calculate $\mathsf{cap}(\mathcal{C})$ for this channel $\mathcal{C}$. [Hint: The channel is symmetric in the sense that the rows are permutations of each other, and the columns are permutations for each other.]

5. Suppose $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$. Consider the random variable $Z = 4 \cdot X + 5 \cdot Y$. Suppose $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$. What is the differential entropy $h(Z)$ of Z? $\boxed{20}$